

THE REES-SUSCHKEWITSCH THEOREM FOR SIMPLE TOPOLOGICAL SEMIGROUPS

TARAS BANAKH, SVETLANA DIMITROVA, AND OLEG GUTIK

ABSTRACT. We detect topological semigroups that are topological paragroups, i.e., are isomorphic to a Rees product $[X \times H \times Y]_\sigma$ of a topological group H over topological spaces X, Y with a continuous sandwich function $\sigma : Y \times X \rightarrow H$. We prove that a simple topological semigroup S is a topological parargroup if one of the following conditions is satisfied: (1) S is completely simple and the maximal subgroups of S are topological groups, (2) S contains an idempotent and the square $S \times S$ is countably compact or pseudocompact, (3) S is sequentially compact or the power $S^{2^{\mathbb{N}}}$ is countably compact. The last item generalizes an old Wallace's result saying that each simple compact topological semigroup is a topological parargroup.

This paper was motivated by the classical Rees-Suschkewitsch Theorem that describes the algebraic structure of completely simple semigroups and the topological versions of this theorem proved for compact topological semigroups by Wallace [28], for compact semitopological semigroups by Ruppert [25], and for sequential countably compact topological semigroups by Gutik, Pagon and Repovš [14]. All topological semigroups considered in this paper are Hausdorff.

We recall that a semigroup S is *simple* if S contains no proper two-sided ideal. A simple semigroup S is called *completely simple* if the set $E = \{e \in S : ee = e\}$ of idempotents of S contains a *primitive idempotent*, that is, a minimal idempotent with respect to the partial order $e \leq f$ on E defined by $ef = fe = e$. In this case all the idempotents are primitive and $H_e = eSe$ is a group for every $e \in E$, see [7, Section 2.7, Ex. 6(b)] or [29].

Let us observe that each group is a completely simple semigroup. Less trivial examples of such semigroups appear as minimal ideals in compact right-topological semigroups, see [25, Theorem I.3.13], [16, Theorem 2.9] and [24]. A generic example of a completely simple semigroup can be constructed as follows. Take any group H and a function $\sigma : Y \times X \rightarrow H$ defined on the product of two sets. This function σ induces the semigroup operation

$$(x, h, y) \cdot (x', h', y') = (x, h\sigma(y, x')h', y')$$

on the product $X \times H \times Y$ turning it into a completely simple semigroup, called the *Rees product of H over X and Y relative to the sandwich map σ* [17] or a *parargroup* [25] and denoted by $[X, H, Y]_\sigma$.

The Rees-Suschkewitsch Structure Theorem [22] says that the converse is also true: each completely simple semigroup S is isomorphic to the parargroup $[X_e, H_e, Y_e]_\sigma$ where e is any idempotent of S , $H_e = eSe$ is the maximal subgroup of S containing

Date: February 24, 2009.

2000 Mathematics Subject Classification. Primary 22A15, 20M20. Secondary 20M18, 54H15.

Key words and phrases. Topological semigroup, semitopological semigroup, Rees-Suschkewitsch Theorem, topological parargroup.

e , $X_e = Se \cap E$, $Y_e = eS \cap E$, and the sandwich function $\sigma : Y_e \times X_e \rightarrow H_e$ is defined by $\sigma(y, x) = yx$. In fact, the map

$$R : [X_e, H_e, Y_e]_\sigma \rightarrow S, \quad R : (x, h, y) \mapsto xhy,$$

is an isomorphism called the *Rees isomorphism*. Its inverse $R^{-1} : S \rightarrow [X_e, H_e, Y_e]_\sigma$ is defined by the formula

$$R^{-1}(s) = (s(ese)^{-1}, ese, (ese)^{-1}s).$$

Now assume that S is a topological semigroup (i.e., a topological space endowed with a continuous semigroup operation). In this case the spaces $X_e = Se \cap E$, $H_e = eSe$, and $Y_e = eS \cap E$ carry the induced topologies while the paragroup $[X_e, H_e, Y_e]_\sigma$ carries the product topology making the semigroup operation continuous, i.e., $[X_e, H_e, Y_e]_\sigma$ is a topological semigroup. Let us observe that the maximal subgroup H_e of S is a paratopological group, which means that the group multiplication on H_e is jointly continuous. If, in addition, the inversion map $x \mapsto x^{-1}$ is continuous on H_e , then H_e is a *topological group*.

Looking at the Rees isomorphism

$$R : [X_e, H_e, Y_e]_\sigma \rightarrow S, \quad R : (x, h, y) \mapsto xhy,$$

we see that it is continuous while its inverse

$$R^{-1} : s \mapsto (s(ese)^{-1}, ese, (ese)^{-1}s)$$

is continuous if the paratopological group H_e is a topological group. In this case the topological semigroup $[X_e, H_e, Y_e]_\sigma$ is called a *topological paragroup* [17].

More precisely, by a *topological paragroup* we understand a topological semigroup that is topologically isomorphic to the Rees product $[X, H, Y]_\sigma$ where H is a topological group and $\sigma : X \times Y \rightarrow H$ is a continuous function defined on the product of two topological spaces.

In such a way we have obtained the following topological Rees-Suschkewitsch structure theorem.

Theorem 1. *A topological semigroup S is a topological paragroup if and only if S is completely simple and each maximal subgroup H_e of S is a topological group.*

There is a simple algebraic characterization of completely simple semigroups, see [7, Theorem 2.54] and [2].

Theorem 2 (Andersen). *A semigroup S is simple if and only if S has an idempotent but contains no isomorphic copy of the bicyclic semigroup $\mathcal{C}(p, q)$.*

We recall that $\mathcal{C}(p, q)$ is a semigroup with a two-sided unit 1, generated by two elements p, q and one relation $qp = 1$.

Combining the Andersen Theorem 2 with Theorem 1 we obtain another characterization of topological paragroups.

Theorem 3. *A topological semigroup S is a topological paragroup if and only if S is simple, contains an idempotent, contains no copy of the bicyclic group, and each maximal subgroup H_e of S is a topological group.*

For compact topological semigroups the last three conditions always are satisfied: such semigroups contain an idempotent by the Iwasawa-Numakura Theorem (see [18, 11, 20, 27] or [6, Vol. 1, Theorem 1.8]), contain no copy of the bicyclic semigroup

by the Koch-Wallace Theorem [19] and the maximal subgroups $H_e = eSe$ corresponding to minimal idempotents are topological groups, being compact paratopological groups, see [9]. In such a way we have proved the following theorem due to Wallace [28].

Theorem 4 (Wallace). *Each simple compact topological semigroup S is topologically isomorphic to a topological paragroup.*

In [25, Theorem I.5.3] the Wallace Theorem was generalized to compact semitopological semigroups. By a *semitopological semigroup* we understand a Hausdorff topological space S endowed with a separately continuous semigroup operation.

Theorem 5 (Ruppert). *Each simple compact semitopological semigroup S is topologically isomorphic to a topological paragroup.*

Another direction of generalization of the Wallace Theorem consists in replacing the compactness assumption by a weaker property. The first step in this direction was made in [14].

Theorem 6 (Gutik-Pagon-Repovš). *Each simple sequential countably compact topological semigroup is a topological paragroup.*

In this paper we generalize both the Wallace and Gutik-Pagon-Repovš Theorems proving that simple topological semigroups satisfying certain compactness-like properties are topological paragroups. All topological spaces considered in this paper are assumed to be Hausdorff.

We recall that a topological space X is

- *countably compact* if each closed discrete subspace of X is finite;
- *pseudocompact* if X is Tychonov and each continuous real-valued function on X is bounded;
- *sequentially compact* if each sequence $\{x_n\}_{n \in \omega} \subset X$ has a convergent subsequence;
- *p -compact* for some free ultrafilter p on ω if each sequence $\{x_n\}_{n \in \omega} \subset X$ has a p -limit $x_\infty = \lim_{n \rightarrow p} x_n$ in X .

Here the notation $x_\infty = \lim_{n \rightarrow p} x_n$ means that for each neighborhood $O(x_\infty) \subset X$ of x_∞ the set $\{n \in \omega : x_n \in O(x_\infty)\}$ belongs to the ultrafilter p . It is clear that each sequentially compact and each compact topological space is p -compact for every ultrafilter p .

By [12], a topological space X is p -compact for some free ultrafilter p on ω if and only if each power X^κ of X is countably compact if and only if the power X^{2^c} is countably compact. It is easy to see that each sequence $(x_n)_{n \in \omega}$ in a countably compact topological space X has p -limit $\lim_{n \rightarrow p} x_n$ for some free ultrafilter p on ω .

We shall say that for some free filter p on ω a double sequence $\{x_{m,n}\}_{m,n \in \omega} \subset X$ has a double p -limit $\lim_{n \rightarrow p} \lim_{m \rightarrow p} x_{m,n}$ if $P = \{n \in \omega : \exists \lim_{m \rightarrow p} x_{m,n} \in X\} \in p$ and the sequence $(\lim_{m \rightarrow p} x_{m,n})_{n \in P}$ has a p -limit in X .

We define a topological space X to be *doubly countably compact* if each double sequence $(x_{m,n})_{m,n \in \omega}$ in X has a double p -limit $\lim_{n \rightarrow p} \lim_{m \rightarrow p} x_{m,n} \in X$ for some free ultrafilter p on ω .

Proposition 1. *A topological space X is doubly countably compact if X is either sequentially compact or p -compact for some free ultrafilter p on ω .*

Proof. The double countable compactness of p -compact spaces is obvious. Now assume that X is sequentially compact and take any double sequence $(x_{m,n})_{m,n \in \omega}$ in X . By the sequential compactness of X there is an infinite subset $A_0 \subset \omega$ such that the subsequence $(x_{m,0})_{m \in A_0}$ converges to some point $x_0 \in X$ in the sense that for each neighborhood $O(x_0) \subset X$ the set $\{n \in A_0 : x_{m,0} \notin O(x_0)\}$ is finite. Now consider the sequence $(x_{m,1})_{m \in A_0}$ and by the sequential compactness of X find an infinite subset $A_1 \subset A_0$ such that the subsequence $(x_{m,1})_{m \in A_1}$ converges to some point x_1 . Next we proceed by induction and for every $n \in \omega$ construct an infinite subset $A_n \subset A_{n-1}$ such that the sequence $(x_{m,n})_{m \in A_n}$ converges to some point $x_n \in S$. Now take any infinite subset $A \subset \omega$ such that for $A \subset^* A_n$ for every $n \in \omega$. The latter means that the complement $A \setminus A_n$ is finite. It follows that for every $n \in \omega$ the sequence $(x_{m,n})_{m \in A}$ converges to the point x_n . By the sequential compactness of S for the sequence $(x_n)_{n \in A}$ there is an infinite subset $B \subset A$ such that the sequence $(x_n)_{n \in B}$ converges to some point $x \in X$. Finally, take any free ultrafilter $p \ni B$ and observe that $x = \lim_{n \rightarrow p} \lim_{m \rightarrow p} x_{m,n}$. \square

Theorem 3 ensures that a simple topological semigroup S is a topological para-group provided

- (1) S has an idempotent;
- (2) S contains no copy of the bicyclic semigroup;
- (3) all maximal subgroups of S are topological groups.

Topological semigroups containing an idempotent can be characterized as follows.

Theorem 7. *A topological semigroup S contains an idempotent if and only if for some $x \in S$ the double sequence $(x^{m-n})_{m \geq n}$ has a double p -limit $\lim_{n \rightarrow p} \lim_{m \rightarrow p} x^{m-n} \in S$ for some free ultrafilter p on ω .*

Proof. The “only if” part is trivial: just take any idempotent x of X and observe that $\lim_{n \rightarrow p} \lim_{m \rightarrow p} x_{m,n} = x$ for any free ultrafilter p on ω .

To prove the “if” part, assume that for some $x \in S$ the double sequence $(x^{m-n})_{m \geq n}$ has a double p -limit $e = \lim_{n \rightarrow p} \lim_{m \rightarrow p} x^{m-n}$ for some free ultrafilter p on ω .

We claim that e is an idempotent. Let $P \in p$ be the set of the numbers n for which there is a p -limit $e_{-n} = \lim_{m \rightarrow p} x^{m-n}$ in S . Then $e = \lim_{P \ni n \rightarrow p} e_{-n}$.

Assuming that e fails to be an idempotent, we can find a neighborhood $O(e) \subset S$ of e such that $O(e) \cdot O(e)$ is disjoint with $O(e)$. Since $e = \lim_{P \ni n \rightarrow p} e_{-n}$, the set

$P_1 = \{n \in P : e_{-n} \in O(e)\}$ belongs to the ultrafilter p .

Take any element $n \in P_1$ and observe that $\lim_{m \rightarrow p} x^{m-n} = e_{-n} \in O(e)$ implies $P_2 = \{m \in P_1 : m > n \text{ and } x^{m-n} \in O(e)\} \in p$. Pick any $m > n$ in P_1 and observe that $\lim_{i \rightarrow p} x^{i-m} = e_{-m} \in O(e)$ and thus the set $P_3 = \{i \in P_2 : i > m \text{ and } x^{i-m} \in O(e)\}$ belongs to p . Now take any number $i \in P_3$ and observe that $i \in P_3 \subset P_2$ and $m \in P_2$ imply $x^{i-n}, x^{i-m}, x^{m-n} \in O(e)$. On the other hand, $x^{i-n} = x^{i-m} x^{m-n} \in O(e) \cdot O(e) \subset S \setminus O(e)$, which is a desired contradiction. \square

This characterization will be applied to obtain some convenient conditions on a topological semigroup X guaranteeing the existence of an idempotent $e \in S$.

Theorem 8. *A topological semigroup S contains an idempotent if S satisfies one of the following conditions:*

- (1) S is doubly countably compact;
- (2) S is sequentially compact;
- (3) S is p -compact for some free ultrafilter p on ω ;
- (4) S^{2^c} is countably compact;
- (5) S^{κ^ω} is countably compact, where κ is the minimal cardinality of a closed subsemigroup of S .

Proof. The first item follows immediately from Theorem 7 and the definition of a doubly sequentially countably compact space.

The next two assertions follow from the first one and Proposition 1. The fourth assertion follows from the third one and the characterization of spaces with countably compact power S^{2^c} as p -compact spaces for some free ultrafilter p , see [12].

It remains to prove the last assertion. Let κ be the smallest cardinality of a closed subsemigroup of S and assume that the power S^{κ^ω} is countably compact. Replacing S by a suitable closed subsemigroup, we can assume that $|S| = \kappa$. Now it suffices to prove that the space S is p -compact for some free ultrafilter p . For every $n \in \omega$ consider the functional $\delta_n : S^\omega \rightarrow S$ assigning to each function $f \in S^\omega$ its value $\delta_n(f) = f(n)$ at n . This functional is an element of the power S^{S^ω} . The countable compactness of S^{S^ω} guarantees that the sequence $(\delta_n)_{n \in \omega}$ has an accumulation point $\delta_\infty \in S^{S^\omega}$ and hence $\delta_\infty = \lim_{n \rightarrow p} \delta_n$ for some free ultrafilter p on ω . Then every function $f \in S^\omega$ has the p -limit

$$\lim_{n \rightarrow p} f(n) = \lim_{n \rightarrow p} \delta_n(f) = \delta_\infty(f),$$

which means that the space S is p -compact. \square

Remark 1. Theorem 8 generalizes many known results related to idempotents in topological semigroups. In particular, it generalizes a result of A. Tomita [26] on the existence of idempotents in p -compact cancellative semigroups as well as the classical Iwasawa-Numakura Theorem [6, Vol.1, Theorem 1.6] on the existence of an idempotent in compact topological semigroups.

The following theorem generalizing both the Wallace and Gutik-Pagon-Repovš Theorems is the main result of this note.

Theorem 9. *A simple topological semigroup S is a topological paragroup if S is doubly countably compact and has countably compact square $S \times S$.*

This theorem follows immediately from Theorem 7 and the following characterizing

Theorem 10. *A topological semigroup S with countably compact square $S \times S$ is a topological paragroup if and only if S is simple and contains an idempotent.*

Proof. The “only if” part is trivial. To prove the “if” part, assume that S is simple and contains an idempotent.

First we check that S contains no copy of $\mathcal{C}(p, q)$. Assume conversely that $\mathcal{C}(p, q) \subset S$ and consider the sequence $\{(q^n, p^n)_{n=1}^\infty\}$ in $\mathcal{C}(p, q) \times \mathcal{C}(p, q) \subset S \times S$. The countable compactness of $S \times S$ guarantees that this sequence has an accumulation point $(a, b) \in S \times S$. Since $q^n p^n = 1$, the continuity of the semigroup operation on S guarantees that $ab = 1$. By Corollary I.2 [8], the bicyclic semigroup $\mathcal{C}(p, q)$ endowed with the topology induced from S is a discrete topological space. So, we can find a neighborhood $O(1) \subset S$ of $1 \in \mathcal{C}(p, q)$ containing no other points of the

semigroup $\mathcal{C}(p, q)$. Since $ab = 1$, the points a, b have neighborhoods $O(a), O(b) \subset S$ such that $O(a) \cdot O(b) \subset O(1)$. Since a is an accumulation point of the sequence q^n , we can find $n \in \mathbb{N}$ with $q^n \in O(a)$. By the same reason, there is a number $m > n$ such that $p^m \in O(b)$. Then $q^n p^m = p^{m-n} \in O(a) \cdot O(b) \cap \mathcal{C}(p, q) = \{1\}$, which is a contradiction. This contradiction shows that the simple semigroup S contains no copy of $\mathcal{C}(p, q)$ and hence is completely simple by the Andersen's Theorem 2.

For each idempotent $e \in E$ the maximal semigroup $H_e = eSe$ is countably compact, being a continuous image of the countably compact space S . Moreover, the square $H_e \times H_e$ is countably compact, being a continuous image of the countably compact space $S \times S$. Then H_e is a topological group, being a paratopological group with countably compact square, see [21] or [1, 2.2]. Now Theorem 3 assures that S is a topological paragroup. \square

For Tychonov topological semigroups the countable compactness of the square $S \times S$ in the preceding theorem can be replaced by its pseudocompactness.

Theorem 11. *A topological semigroup S with pseudocompact square $S \times S$ is a topological paragroup if and only if S is simple and contains an idempotent.*

Proof. The “only if” part of the theorem is trivial. To prove the “if” part, assume that the square $S \times S$ is pseudocompact. By [3, 1.3], the Stone-Ćech compactification βS of S is a compact topological semigroup. By the Koch-Wallace Theorem [19], the topological semigroup βS , being compact, contains no isomorphic copy of the bicyclic semigroup and consequently, and so does the subsemigroup S of βS . By the Andersen Theorem [7, Theorem 2.54], the simple semigroup S is completely simple. In order to apply Theorem 1, it remains to prove that each maximal subgroup H_e of S is a topological group. Since the idempotent e of S is primitive, the maximal group H_e coincides with eSe and hence pseudocompact, being the continuous image of the pseudocompact space S . Applying Theorem 2.6 of [23], we conclude that H_e , being a pseudocompact paratopological group, is a topological group. \square

For completely simple semigroups S the pseudocompactness of the square $S \times S$ in the preceding theorem can be replaced by the pseudocompactness of S .

Theorem 12. *A pseudocompact topological semigroup S is a topological paragroup if and only if S is completely simple.*

Proof. The “only if” part is trivial. To prove the “if” part, assume that S is a completely simple pseudocompact topological semigroup. Take any primitive idempotent e of S and observe that the maximal subgroup $H_e = eSe$ is pseudocompact, being the continuous image of the pseudocompact space S . By Theorem 2.6 of [23], the paratopological group H_e , being pseudocompact, is a topological group. Applying Theorem 1, we conclude that S is a topological paragroup. \square

Our final result describes the structure of simple sequential countably compact topological semigroups. We recall that a topological space X is called *sequential* if for each non-closed subset $A \subset X$ there is a sequence $\{a_n\}_{n \in \omega} \subset A$ that converges to some point $x \in X \setminus A$.

Theorem 13. *For a simple topological semigroup S the following conditions are equivalent:*

- (1) *S is a regular sequential countably compact topological space;*

- (2) S is topologically isomorphic to a topological paragroup $[X, G, Y]_\sigma$ for some regular sequential countably compact topological spaces X and Y and a sequential countably compact topological group G .

Proof. (1) \Rightarrow (2). Assume that S is a regular sequential countably compact topological space. It follows that S is sequentially compact. By Theorem 9, S is topologically isomorphic to a topological paragroup $[X, G, Y]_\sigma$ for some topological spaces X and Y and some topological group G . The spaces X, Y, G , being homeomorphic to closed subspaces of S , are regular sequential and countably compact.

(2) \Rightarrow (1) Assume that S is topologically isomorphic to a topological paragroup $[X, G, Y]_\sigma$ for some regular sequential countably compact topological spaces X, Y and some sequential countably compact topological group G . It is clear that those spaces are sequentially compact. It follows from Boehme Theorem [10, 3.10.J(c)] that the product $X \times G \times Y$ is sequential. Since the product of sequentially compact spaces is sequentially compact (and hence countably compact), the space S , being homeomorphic to $X \times G \times Y$, is regular sequential and countably compact. \square

OPEN PROBLEMS

Problem 1. *Let X be a doubly countably compact space. Is the square $X \times X$ countably compact?*

Problem 2. *Let S be a (simple) semitopological semigroup with countably compact power S^c . Has S an idempotent?*

Problem 3. *Assume that a simple Tychonov countably compact topological semigroup S contains an idempotent. Is S completely simple? Equivalently, is S a topological paragroup?*

It is known that each (topological) semigroup embeds into a simple (topological) semigroup, see [5], [7, Theorem 8.45] and [13].

Problem 4. *Is it true that each countably compact topological semigroup embeds into a simple countably compact topological semigroup?*

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DEPARTMENT OF MATHEMATICS, LVIV NATIONAL UNIVERSITY, UNIVERSYTETSKA 1, LVIV, 79000, UKRAINE

E-mail address: tbanakh@yahoo.com

NATIONAL TECHNICAL UNIVERSITY "KHARKOV POLYTECHNICAL INSTITUTE", FRUNZE 21, KHARKIV, 61002, UKRAINE

DEPARTMENT OF MECHANICS AND MATHEMATICS, IVAN FRANKO LVIV NATIONAL UNIVERSITY, UNIVERSYTETSKA 1, LVIV, 79000, UKRAINE

E-mail address: o_gutik@franko.lviv.ua